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Path integral treatment of the hydrogen atom in a curved space of constant curvature

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Abstract. The path integral treatment of the hydrogen atom in a spherical space is discussed. The dynamical group SU(1, 1) of the system is used for path integration. By mapping the radial path integral onto the SU(1, 1) manifold, the energy spectrum and the normalised wavefunctions are obtained. In the flat space limit, the standard hydrogen spectrum and the corresponding normalised energy eigenfunctions are recovered. The scattering states are also found in the limit.

1. Introduction

Quantum mechanics of the Kepler problem in a spherical space was first solved by Schrödinger [1]. In fact, he considered this problem as an example that can be handled by the factorisation method but is difficult to tackle in any other way. Soon after, however, Stevenson [2] succeeded in finding the solution by a conventional method. Recently, an algebraic method has also been used to obtain the solution by recognising that the system has an SU(1, 1) dynamical symmetry [3]. The purpose of the present paper is to report that the Kepler problem in the curved space can also be solved by path integration if the integration is done over paths in the dynamical group manifold.

It is generally true that solving Schrödinger's equation for a given system is easier than calculating Feynman's path integral for the same system. However, in recent years, there have been considerable developments in path integration techniques. It has been seen to be particularly useful to incorporate the dynamical symmetry into the path integral [4–8]. In fact, the dynamical symmetry is more than an heuristic guide in constructing a path integral. The dynamical group manifold presents itself as an arena on which path integration may explicitly be carried out. The Kepler problem in a spherical space as treated here is indeed an example that is path integrable in the dynamical group manifold. The present path integral calculated for the Kepler problem not only suggests a link underlying the methods of factorisation, algebraisation and path integration but also indicates that the path integral method has become capable of handling a problem such as the one Schrödinger once considered difficult to tackle.

In the flat-space limit $R \to \infty$ our solution coincides with the well known result for the hydrogen-like atom. Although the path integral treatments of the hydrogen atom are available [9, 10], the present calculation provides another path integral approach to the hydrogen problem in flat space.

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In § 2, we construct a path integral pertinent to the Kepler problem in a spherical space. Then in § 3 we convert the result into an SU(1, 1) path integral to find the energy-dependent Green function for the Kepler problem. Section 4 deals with the resultant energy spectrum, the normalised wavefunctions and their flat-space limits.

2. Path integral for the Coulomb problem in spherical space

The spherical space in question is a uniformly curved space with a positive curvature. Let the curvature be $K=1/R^2>0$. Then the line element $\mathrm{d}s$ of the space is given in polar coordinates by

$$\mathrm{d}s^2=f(r)\,\mathrm{d}r^2+r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2) \tag{2.1}$$

where $f(r)=(1-r^2/R^2)^{-1}$. It can also be put in the form

$$\mathrm{d}s^2=R^2\,\mathrm{d}\chi^2+R^2\sin^2\chi(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2) \tag{2.2}$$

where $\sin\chi=r/R$, $\chi\in[0,\pi]$. The Lagrangian for the Coulomb problem in this space is

$$L=\tfrac{1}{2}M\dot{s}^2+(Ze^2/R)\cot\chi \tag{2.3}$$

where M is the mass of a particle moving in the central potential $V=-(Ze^2/r)(1-r^2/R^2)^{1/2}$.

For this system we consider the following path integral:

$$P(\boldsymbol{r}'',\boldsymbol{r}';\tau)=\int\exp\left(\frac{\mathrm{i}}{\hbar}\int(L+E)\,\mathrm{d}t\right)\mathrm{D}^3\boldsymbol{r}(t) \tag{2.4}$$

which is the object referred to earlier as the promoter [11]. From (2.4), the energy-dependent function $G(\boldsymbol{r}'',\boldsymbol{r}';E)$ and the propagator $K(\boldsymbol{r}'',\boldsymbol{r}';t''-t')$ can be evaluated, respectively, by

$$G(\boldsymbol{r}'',\boldsymbol{r}';E)=\frac{1}{\mathrm{i}\hbar}\int P(\boldsymbol{r}'',\boldsymbol{r}';\tau)\,\mathrm{d}\tau \tag{2.5}$$

$$K(\boldsymbol{r}'',\boldsymbol{r}';t''-t')=\frac{1}{2\pi\hbar}\int\int P(\boldsymbol{r}'',\boldsymbol{r}';\tau)\exp\left(-\frac{\mathrm{i}}{\hbar}E(t''-t')\right)\mathrm{d}\tau\,\mathrm{d}E \tag{2.6}$$

with $t''>t'$. The polar coordinate representation of the path integral (2.4) in the time-sliced version is

$$P(\boldsymbol{r}'',\boldsymbol{r}';\tau)=\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left(\frac{\mathrm{i}}{\hbar}W_j\right)\prod_{j=1}^{N}\left(\frac{M}{2\pi\mathrm{i}\hbar\tau_j}\right)^{3/2}$$
$$\times\prod_{j=1}^{N-1}\tfrac{1}{2}R^3\sin^2\chi_j\,\mathrm{d}\chi_j\sin\theta_j\,\mathrm{d}\theta_j\,\mathrm{d}\phi_j. \tag{2.7}$$

Note that as $\chi\in[0,\pi]$ the range of r is covered twice and therefore a factor $\frac{1}{2}$ has been included in the measure of (2.7). Here W_j is the short-time action (Hamilton's characteristic function for a short-time interval τ_j) given by

$$W_j=\int_{t_{j-1}}^{t_j}(L+E)\,\mathrm{d}t=\frac{M}{2\tau_j}(\Delta s_j)^2+\frac{Ze^2}{R}\tau_j\cot\chi_j+E\tau_j. \tag{2.8}$$

As usual, we have set $\boldsymbol{r}_j = \boldsymbol{r}(t_j)$, $t' = t_0$, $t'' = t_N$ and $\tau_j = t_j - t_{j-1}$. Recall that in path integration all terms of $O(\tau_j^{1+\varepsilon})$ with $\varepsilon > 0$ may be ignored and that $(\Delta q_j)^2 \sim \tau_j$ for any coordinate variable q_j. For a polar coordinate path integral in flat space, it is known that the terms $(\Delta\theta)^4$ and $(\Delta\phi)^4$ resulting from the expansion of $(\Delta \boldsymbol{r}_j)^2$ cannot be neglected. This is because $(\Delta\theta)^4/\tau_j$ and $(\Delta\phi)^4/\tau_j$ appearing in the kinetic term are of $O(\tau_j)$. For a path integral in curved space, the book-keeping rules remain the same. However, the metric relation (2.1) holds only locally and cannot arbitrarily be extended to a finite interval. Therefore special care is needed. The sliced-time version of (2.2) for an n-dimensional sphere S^n has been shown to be [5]:

$$(\Delta s_j)^2 = 2R^2(1-\cos\omega_j) + n(n-2)\hbar^2\tau_j^2/4M^2R^2 \tag{2.9}$$

where $\cos\omega_j = \boldsymbol{q}_j \cdot \boldsymbol{q}_{j-1}$. For $n=3$, $\boldsymbol{q}$ is given in polar coordinates as

$$\boldsymbol{q} = \begin{pmatrix} \sin\chi\sin\theta\sin\phi \\ \sin\chi\sin\theta\cos\phi \\ \sin\chi\cos\theta \\ \cos\chi \end{pmatrix} \tag{2.10}$$

and hence

$$\cos\omega_j = \cos\Delta\chi_j - \sin\chi_j \sin\chi_{j-1}(1-\cos\Theta_j)$$

with $\cos\Theta_j = \cos\theta_j\cos\theta_{j-1} + \sin\theta_j\sin\theta_{j-1}\cos\Delta\phi_j$. The short-time action (2.8) becomes

$$W_j = \frac{MR^2}{\tau_j}(1-\cos\Delta\chi_j) + \frac{MR^2}{\tau_j}\sin\chi_j\sin\chi_{j-1}(1-\cos\Theta_j) + \frac{Ze^2}{R}\tau_j\cot\chi_j + \left(E + \frac{3\hbar^2}{8MR^2}\right)\tau_j. \tag{2.11}$$

Using the approximation $(1-\cos\Delta\chi_j) \approx \frac{1}{2}(\Delta\chi_j)^2 - \frac{1}{24}(\Delta\chi_j)^4$ valid in path integration and the formula for large a, $\operatorname{Re} a > 0$,

$$\int_0^\infty x^{2n}\exp(-ax^2 + bx^4 + O(x^6))\,dx \approx \int_0^\infty x^{2n}\exp(-ax^2 + 3b/4a^2 + O(a^{-3}))\,dx \tag{2.12}$$

we may replace the fourth-order term $(\Delta\chi_j)^4$ by an equivalent one; namely,

$$-(MR^2/24\tau_j)(\Delta\chi_j)^4 \sim \hbar^2\tau_j/8MR^2.$$

Therefore the short-time action (2.11) can effectively be written as

$$W_j = \frac{MR^2}{2\tau_j}(\Delta\chi_j)^2 + \frac{MR^2}{\tau_j}\widehat{\sin^2}\chi_j(1-\cos\Theta_j) + \frac{Ze^2}{R}\tau_j\cot\chi_j + (E + \hbar^2/2MR^2)\tau_j \tag{2.13}$$

where we have set $\widehat{\sin^2}\chi_j = \sin\chi_j\sin\chi_{j-1}$. The energy shift by $\hbar^2/2MR^2$, seen in (2.13), has also been obtained for a free particle in a space of constant curvature from a different aspect [12]. By employing the asymptotic expansion for large $|z|$

$$\exp[iz(1-\cos\Theta_j)] = \frac{i}{2z}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(2l+1)\frac{(l-m)!}{(l+m)!}\exp(im\Delta\phi_j)P_l^m\cos\theta_j \times P_l^m(\cos\theta_{j-1})\exp[-il(l+1)/2z] \tag{2.14}$$

we carry out the integration over θ_j and ϕ_j of (2.7) to separate variables as [13]:

$$P(\boldsymbol{r}'', \boldsymbol{r}'; \tau) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} P_l(r'', r'; \tau) Y_l^{m*}(\theta', \phi') Y_l^m(\theta'', \phi'') \tag{2.15}$$

where

$$P_l(r'', r'; \tau) = (R^3 \sin\chi' \sin\chi'')^{-1} \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left(\frac{\mathrm{i}}{\hbar} W_j\right) \prod_{j=1}^{N} \left(\frac{MR^2}{2\pi \mathrm{i}\hbar\tau_j}\right)^{1/2} \prod_{j=1}^{N-1} \tfrac{1}{2}\,\mathrm{d}\chi_j \tag{2.16}$$

with

$$W_j = \frac{MR^2}{2\tau_j}(\Delta\chi_j)^2 - \frac{l(l+1)\hbar^2}{2MR^2\,\widehat{\sin}^2\chi_j}\tau_j + \frac{Ze^2}{R}\tau_j \cot\chi_j + \left(E + \frac{\hbar^2}{2MR^2}\right)\tau_j. \tag{2.17}$$

In this manner, we have constructed a path integral pertinent to the Coulomb problem in a spherical space. Our next task is to evaluate the radial path integral (2.16). It is certainly difficult to perform the path integration of (2.16) in the standard path integral technique.

3. Realisation of the dynamical symmetry group by an SU(1, 1) path integral

To tame the wild-looking path integral (2.16), we make use of the dynamical symmetry SU(1, 1) discussed for the algebraisation of the same Coulomb problem [3]. First, as in [3], we transform the variable χ into a new one β by

$$\mathrm{e}^{\beta} = \tanh(\mathrm{i}\chi/2) \tag{3.1}$$

from which follows

$$(\Delta\chi_j)^2 = -\widehat{\mathrm{cosech}}^2\,\beta_j[(\Delta\beta_j)^2 + \tfrac{1}{2}(1 - \widehat{\mathrm{cosech}}^2\,\beta_j)(\Delta\beta_j)^4].$$

Since (3.1) is a complex transformation, we are actually making an analytical continuation of (2.16) into the domain where $0 \leqslant \mathrm{Re}\,\chi \leqslant \pi$ and $-\infty < \mathrm{Im}\,\chi \leqslant 0$ so that $\beta \in (-\infty, +\infty)$. At the same time, we also complexify Ze^2/R by setting

$$\kappa = Ze^2/\mathrm{i}R \tag{3.2}$$

to find $(Ze^2/R)\cot\chi = -\kappa\cosh\beta$. In addition to the change of variables, we rescale each local time interval τ_j of (2.16) into σ_j by [11]:

$$\sigma_j = \tfrac{1}{4}\tau_j\,\widehat{\sinh}^2\,\beta_j \tag{3.3}$$

such that

$$\sigma = \tfrac{1}{4}\tau \sinh\beta' \sinh\beta'' \tag{3.4}$$

where $\sigma = \sum \sigma_j$ and $\tau = \sum \tau_j$.

After these transformations are made, the short-time action (2.17) is

$$W_j = \frac{MR^2}{\sigma_j}\left[-\frac{1}{2}\left(\frac{\Delta\beta_j}{2}\right)^2 - \frac{1}{24}\left(\frac{\Delta\beta_j}{2}\right)^4\right] - \frac{MR^2}{2\sigma_j}\left(\frac{1}{4} - \frac{1}{3\,\widehat{\sinh}^2\,\beta_j}\right)\left(\frac{\Delta\beta_j}{2}\right)^4$$
$$+ \frac{2l(l+1)\hbar^2}{MR^2}\sigma_j + 4\sigma_j\,\widehat{\mathrm{cosech}}^2\,\beta_j\left(E + \frac{\hbar^2}{2MR^2} - \kappa\cosh\beta_j\right). \tag{3.5}$$

The second term of (3.5), which contains $(\Delta\beta_j)^4$, can be replaced as before with the aid of (2.12) by $(\hbar^2\sigma_j/2MR^2)(\frac{3}{4}-\widehat{\operatorname{cosech}}^2\beta_j)+O(\sigma_j^2)$. Furthermore, noticing that

$$(\sin\chi'\sin\chi'')^{-1}\prod_{j=1}^{N}\left(\frac{MR^2}{2\pi i\hbar\tau_j}\right)^{1/2}\prod_{j=1}^{N-1}\tfrac{1}{2}\,d\chi_j$$

$$=-(\sinh\beta'\sin\beta'')^{3/2}\prod_{j=1}^{N}\left(\frac{MR^2}{8\pi i\hbar\sigma_j}\right)^{1/2}\prod_{j=1}^{N-1}\tfrac{1}{2}i\,d\beta_j \tag{3.6}$$

we write the radial path integral (2.16) as

$$P_l(r'',r';\tau)=-R^{-3}(\sinh\beta'\sinh\beta'')^{3/2}$$

$$\times\lim_{N\to\infty}\int_{-\infty}^{+\infty}\prod_{j=1}^{N}\exp\left(\frac{i}{\hbar}W_j\right)\prod_{j=1}^{N}\left(\frac{MR^2}{8\pi i\hbar\sigma_j}\right)^{1/2}\prod_{j=1}^{N-1}\tfrac{1}{2}i\,d\beta_j \tag{3.7}$$

where

$$W_j=\frac{MR^2}{\sigma_j}[1-\cosh(\Delta\beta_j/2)]+\frac{(2l+1)^2-\frac{1}{4}}{2MR^2}\hbar^2\sigma_j+\frac{E'-\kappa}{\widehat{\sinh}^2(\beta_j/2)}\sigma_j-\frac{E'+\kappa}{\widehat{\cosh}^2(\beta_j/2)}\sigma_j. \tag{3.8}$$

In the above we have set $E'=E+3\hbar^2/8MR^2$.

The resultant path integral (3.7) is by no means simpler than (2.16) in appearance. However, we notice its similarity in structure to that for the modified Pöschl-Teller potential which has been reduced to an SU(1, 1) path integral [6]. In fact, the reduction procedure used for the Pöschl-Teller potential is basically the same as that devised for converting the Rosen-Morse oscillator into a free particle in an SU(2) manifold [4, 5]. Here we follow the same procedure.

By employing the asymptotic relation valid for large $|z|$ and integer n

$$\exp[-(n^2-\tfrac{1}{4})/2z]=(z/2\pi)^{1/2}\int_0^{2\pi}\exp[in\phi-z(1-\cos\phi)]\,d\phi \tag{3.9}$$

we introduce two additional angular variables ξ and η as

$$\exp\left(-\frac{i}{\hbar}\frac{\kappa+E'}{\widehat{\cosh}^2(\beta_j/2)}\sigma_j\right)=\left(\frac{MR^2\,\widehat{\cosh}^2(\beta_j/2)}{2\pi i\hbar\sigma_j}\right)^{1/2}$$

$$\times\int_0^{2\pi}\exp[ip\Delta\xi_j+(iMR^2/\hbar^2)\,\widehat{\cosh}^2(\beta_j/2)(1-\cos\Delta\xi_j)]\,d\xi_j \tag{3.10}$$

$$\exp\left\{-\frac{i}{\hbar}\frac{\kappa-E'}{\widehat{\sinh}^2(\beta_j/2)}\sigma_j\right\}=\left(-\frac{MR^2\,\widehat{\sinh}^2(\beta_j/2)}{2\pi i\hbar\sigma_j}\right)^{1/2}$$

$$\times\int_0^{2\pi}\exp[iq\Delta\eta_j-(iMR^2/\hbar^2)\,\widehat{\sinh}^2(\beta_j/2)(1-\cos\Delta\eta_j)]\,d\eta_j \tag{3.11}$$

where we have set

$$p=\left(\frac{2MR^2}{\hbar^2}(E+\kappa)+1\right)^{1/2}\qquad q=\left(\frac{2MR^2}{\hbar^2}(E-\kappa)+1\right)^{1/2} \tag{3.12}$$

both of which are assumed to be positive integers. Furthermore, we change the newly introduced variables ξ_j and η_j into Euler angles α_j and γ_j by

$$\alpha_j = \xi_j - \eta_j \qquad \gamma_j = \xi_j + \eta_j \tag{3.13}$$

and

$$\int_0^{2\pi} d\xi_i \int_0^{2\pi} d\eta_j = \frac{1}{2}\int_0^{2\pi} d\alpha_j \int_{-2\pi}^{2\pi} d\gamma_j \tag{3.14}$$

with $\alpha' = \gamma' = 0$. Substitution of (3.10)-(3.14) into (3.7) results in

$$P_l(r'', r'; \tau) = (\mathrm{i}/2R)^3(\sinh\beta' \sinh\beta'')^2 \exp\left(\frac{\mathrm{i}\hbar\sigma}{2MR^2}[(2l+1)^2 - \tfrac{1}{4}]\right)$$
$$\times \int_0^{2\pi} d\alpha'' \int_{-2\pi}^{2\pi} d\gamma'' \exp\left\{\frac{\mathrm{i}}{2}(p-q)\alpha'' + \frac{\mathrm{i}}{2}(p+q)\gamma''\right\} Q(\beta'', \beta'; \alpha''; \gamma''; \sigma). \tag{3.15}$$

Here the function $Q(\beta'', \beta'; \alpha''; \gamma''; \sigma)$ appearing in (3.15) is an SU(1, 1) path integral expressed in terms of Euler variables (α, β, γ). Namely,

$$Q(\beta'', \beta'; \alpha''; \gamma''; \sigma)$$
$$= \lim_{N\to\infty} \int_{\mathrm{SU}(1,1)} \prod_{j=1}^{N} \exp\left(\frac{\mathrm{i}}{\hbar} S_j\right) \prod_{j=1}^{N} \left(\frac{MR^2}{2\pi\mathrm{i}\hbar\sigma_j}\right)^{1/2}$$
$$\times \left(\frac{\mathrm{i}MR^2}{2\pi\hbar\sigma_j}\right) \prod_{j=1}^{N-1} \tfrac{1}{8} \sinh\beta_j \, d\beta_j \, d\alpha_j \, d\gamma_j \tag{3.16}$$

with

$$S_j = \frac{MR^2}{\sigma_j}[1 - \cosh(\Omega_j/2)] \tag{3.17}$$

and

$$\cosh(\Omega_j/2) = \widehat{\cosh}^2(\beta_j/2)\cos[(\Delta\alpha_j + \Delta\gamma_j)/2] - \widehat{\sinh}^2(\beta_j/2)\cos[(\Delta\alpha_j - \Delta\gamma_j)/2] \tag{3.18}$$

where we have changed the integration over $\beta \in (-\infty, +\infty)$ into two integrations over $\beta \in [0, \infty)$ as the integrand is symmetric in $\beta \to -\beta$. In this fashion, we have completed the conversion of the radial path integral (2.16) into a path integral on the SU(1, 1) group manifold.

4. Performing the SU(1, 1) path integral

The SU(1, 1) path integral (3.16) has been evaluated and given by [6]:

$$Q(\beta'', \beta'; \alpha''; \gamma''; \sigma) = \frac{1}{2\pi^2} \sum_{\lambda=\pm} \sum_{2J=0}^{\infty} (2J+1) \exp[-(\mathrm{i}\sigma/\hbar)C_J]\chi_J^{(\lambda)}(g''g'^{-1})$$
$$+ \frac{1}{2\pi^2} \sum_{\lambda=0,1/2} \int_0^{\infty} ds\, 2s \tanh[\pi(s + \mathrm{i}\lambda)]$$
$$\times \exp\{-(\mathrm{i}\sigma/\hbar)C_{-1/2+\mathrm{i}s}\}\chi_{-1/2+\mathrm{i}s}^{(\lambda)}(g''g'^{-1}) \tag{4.1}$$

where $C_J = (\hbar^2/2MR^2)[(2J+1)^2 - \frac{1}{4}]$ and

$$\chi_J^{(\lambda)}(g''g'^{-1}) = \sum_{\mu,\nu} \exp(-\mathrm{i}\mu\alpha'')\exp(-\mathrm{i}\nu\gamma'')\, V^J_{\mu,\nu}(\beta'')\, V^{J*}_{\mu,\nu}(\beta') \tag{4.2}$$

which are the SU(1, 1) group characters belonging to the unitary irreducible representation of the fundamental series,

$$D_J^{(\lambda)}: \begin{cases} J = -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots & \begin{cases}\mu = J+1, J+2, \ldots & \text{for } \lambda = + \\ \mu = -J-1, -J-2, \ldots & \text{for } \lambda = - \end{cases} \\ J = -\frac{1}{2} + \mathrm{i}s \quad \begin{cases} s \geqslant 0 \\ s > 0 \end{cases} & \begin{matrix} \mu = 0, \pm 1, \pm 2, \ldots & \text{for } \lambda = 0 \\ \mu = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots & \text{for } \lambda = \frac{1}{2}. \end{matrix} \end{cases}$$

The Bargmann function $V^J_{\mu\nu}(\beta)$ used in (4.2) has the property $V^J_{\mu\nu}(\beta) = (-1)^{\mu-\nu} V^J_{\nu\mu}(\beta)$ and may be given explicitly by hypergeometric functions [14, 15]. For example we have for $\lambda = +$

$$V^J_{\mu,\nu}(\beta) = N^J_{\mu\nu}[\cosh(\beta/2)]^{-\mu-\nu}[\sinh(\beta/2)]^{\mu-\nu} \times {}_2F_1(1-\nu+J, -\nu-J; 1+\mu+\nu; -\sinh^2(\beta/2)) \tag{4.3}$$

with

$$N^J_{\mu\nu} = \frac{1}{\Gamma(\mu-\nu+1)}\left(\frac{\Gamma(1+\mu+J)\Gamma(\mu-J)}{\Gamma(1+\nu+J)\Gamma(\nu-J)}\right)^{1/2}.$$

By integration of (4.1) over α'' and γ'', the radial promoter (3.15) becomes, using the above property of the Bargmann function,

$$\begin{aligned} P_l(r'', r'; \tau) = {} & (\mathrm{i}/R)^3(\sinh\beta' \sinh\beta'')^2 \\ & \times \Bigg[\sum_{J=\lambda}^{J_0} (J+\tfrac{1}{2})\, V^{J*}_{(p+q)/2,(p-q)/2}(\beta')\, V^J_{(p+q)/2,(p-q)/2}(\beta'') \\ & \times \exp\left(-\frac{2\mathrm{i}\hbar\sigma}{MR^2}(J^2+J-l^2-l)\right) \\ & + \int_0^\infty \mathrm{d}s\, s \tanh[\pi(s+\mathrm{i}\lambda)]\, V^{-1/2+\mathrm{i}s*}_{(p+q)/2,(p-q)/2}(\beta')\, V^{-1/2+\mathrm{i}s}_{(p+q)/2,(p-q)/2}(\beta'') \\ & \times \exp\left(+\frac{2\mathrm{i}\hbar\sigma}{MR^2}(s^2+l^2+l+\tfrac{1}{4})\right)\Bigg] \end{aligned} \tag{4.4}$$

where $J_0 + 1 = \min(\frac{1}{2}|p-q|, \frac{1}{2}|p+q|)$ and $\lambda = 0$ ($\frac{1}{2}$) for $2J_0$ even (odd). From (3.12) it is obviously that

$$J_0 + 1 = \tfrac{1}{2}(p-q) > 0 \tag{4.5}$$

and therefore the $D_J^{(+)}$ series has been selected in (4.4).

The radial energy function can be evaluated via (2.5) as

$$G_l(r'', r'; E) = (\mathrm{i}\hbar)^{-1}\int_{-\infty}^{\infty} P_l(r'', r'; \tau)\left(\frac{\mathrm{d}\tau}{\mathrm{d}\sigma}\right)\mathrm{d}\sigma. \tag{4.6}$$

Note that $\mathrm{d}\tau/\mathrm{d}\sigma = 4/(\sinh\beta''\sinh\beta')$ and

$$\int_{-\infty}^{\infty}\exp\left(-\mathrm{i}\frac{2\hbar\sigma}{MR^2}(J^2+J-l^2-l)\right)\mathrm{d}\sigma = \frac{\pi MR^2}{\hbar(2J+1)}[\delta(J-l) + \delta(J+l+1)]. \tag{4.7}$$

As l is a positive integer, only the discrete series with $J = l$ contributes to (4.6).

$$G_l(r'', r'; E) = -\frac{2\pi M}{\hbar^2 R} \sinh\beta' \sinh\beta'' \sum_{n_r=0}^{J_0} \delta(J_0 - n_r - l)$$
$$\times V^{l*}_{(p+q)/2, J_0+1}(\beta') V^{l}_{(p+q)/2, J_0+1}(\beta''). \tag{4.8}$$

In (4.8) we have shifted the summation over J into that over $n_r = J_0 - J$. The poles of (4.8) correspond to the energy eigenvalues of the system. From $J_0 = n_r + l$ we have $\frac{1}{2}(p-q) = n$, where $n = n_r + l + 1$. Using (3.12) we find the energy spectrum

$$E_n = \frac{\hbar^2}{2MR^2}(n^2 - 1) - \frac{MZ^2e^4}{2\hbar^2 n^2} \tag{4.9}$$

where $n = 1, 2, 3, \ldots$ and κ has been changed back to $Ze^2/\mathrm{i}R$ by (3.2). This spectrum coincides with that obtained from other methods [1-3].

The corresponding energy eignfunctions can be found by calculating the radial propagator

$$K_l(r'', r'; t'' - t') = \frac{\mathrm{i}}{2\pi} \int_{-\infty}^{\infty} G_l(r'', r'; E) \exp\left(-\frac{\mathrm{i}}{\hbar} E(t'' - t')\right) \mathrm{d}E. \tag{4.10}$$

Note that $\delta(f(E)) = \delta(E - E_n)/|f'(E_n)|$, where E_n is given by (4.9) and $f(E) = J_0(E) - n$. With (3.12) we obtain $|f'(E_n)| = MR^2 n/\hbar^2(\varepsilon_n^2 + n^2)$, where $\varepsilon_n = -R/an$ and $a = \hbar^2/MZe^2$. The integration (4.10) yields

$$K_l(r'', r'; t'' - t') = \sum_{n=l+1}^{\infty} R^*_{nl}(r') R_{nl}(r'') \exp\left(-\frac{\mathrm{i}}{\hbar} E_n(t'' - t')\right) \tag{4.11}$$

with the normalised wavefunctions

$$R_{nl}(r) = \left(\frac{\mathrm{i}(n^2 + \varepsilon_n^2)}{R^3 n}\right)^{1/2} (\mathrm{i}\sinh\beta) V^l_{\mathrm{i}\varepsilon_n, n}(\beta). \tag{4.12}$$

Note that $\frac{1}{2}(p+q) = \mathrm{i}\varepsilon_n$.

The Bargmann function appearing in (4.12) is for $\lambda = +$ and given by (4.3). With the aid of the following formula:

$${}_2F_1(a, b; c; z) = \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(c-a)\Gamma(b)} (-z)^{-a} {}_2F_1(a, a-c+1; a-b+1; 1/z)$$
$$+ \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(c-b)\Gamma(a)} (-z)^{-b} {}_2F_1(b, b-c+1; b-a+1; 1/z)$$

and using the relation

$$\left.\frac{\Gamma(z)}{\Gamma(z-n_r)}\right|_{z=-2l-1} = (-1)^{n_r} \frac{\Gamma(n+l+1)}{\Gamma(2l+2)}$$

we can express the Bargmann function in (4.12) as

$$V^l_{\mathrm{i}\varepsilon_n, n}(\beta) = \frac{(-1)^{n_r}}{\Gamma(2l+2)} \left(\frac{\Gamma(1+l+\mathrm{i}\varepsilon_n)\Gamma(1+l+n)}{\Gamma(\mathrm{i}\varepsilon_n - l)\Gamma(n-l)}\right)^{1/2} [\cosh(\beta/2)]^{-n-\mathrm{i}\varepsilon_n}$$
$$\times [\sinh(\beta/2)]^{n+\mathrm{i}\varepsilon_n - 2l - 2}$$
$$\times {}_2F_1(1-n+l, l-\mathrm{i}\varepsilon_n+1; 2l+2; -\sinh^{-2}(\beta/2)). \tag{4.13}$$

Transforming β back to χ by (3.1) it follows that

$$\sin\chi = \frac{1}{\mathrm{i}\sinh\beta} \qquad \sinh^2(\beta/2) = \frac{\mathrm{e}^{-\mathrm{i}\chi}}{2\mathrm{i}\sin\chi} \qquad \cosh^2(\beta/2) = \frac{\mathrm{e}^{\mathrm{i}\chi}}{2\mathrm{i}\sin\chi}$$

we can write the radial wavefunction (4.11) in the form,

$$R_{nl}(r) = N_{nl}\sin^l\chi\,\exp[-\mathrm{i}\chi(n+\mathrm{i}\varepsilon_n - l - 1)] \times {}_2F_1(1-n+l, 1-\mathrm{i}\varepsilon_n + l; 2l+2; 1-\mathrm{e}^{2\mathrm{i}\chi}) \tag{4.14}$$

with the correct normalisation factor

$$N_{nl} = \exp[\mathrm{i}\pi(l+1+2n_r)/2]\frac{2^{l+1}}{\Gamma(2l+2)}\left(\frac{\mathrm{i}}{R^3}\frac{(n^2+\varepsilon_n^2)}{n}\frac{\Gamma(1+\mathrm{i}\varepsilon_n+l)\Gamma(1+n+l)}{\Gamma(\mathrm{i}\varepsilon_n - l)\Gamma(n-l)}\right)^{1/2} \tag{4.15}$$

where $r = R\sin\chi$. This solution is identical to that given by others apart from the normalisation factor [1-3].

In the flat-space limit $R\to\infty$, the energy spectrum (4.9) for finite n goes over to the well known formula, $E_n = -MZ^2e^4/2\hbar^2n^2$. However, for a large n, comparable with R, so that $n = kR$ (k = constant), we obtain a continuous spectrum, $E = \hbar^2k^2/2M$. By using the following limiting values:

$$\lim_{R\to\infty} {}_2F_1(1-n+l, 1-\mathrm{i}\varepsilon_n+l; 2l+2; 1-\mathrm{e}^{2\mathrm{i}\chi}) = {}_1F_1(1-n+l; 2l+2; 2r/an)$$

$$\lim_{R\to\infty} \exp[-\mathrm{i}\chi(n+\mathrm{i}\varepsilon_n-l-1)] = \exp(-r/an) \tag{4.16}$$

$$\lim_{R\to\infty} \sin^l\chi\left(\frac{\mathrm{i}}{R^3}\frac{(n^2+\varepsilon_n^2)}{n}\frac{\Gamma(1+l+\mathrm{i}\varepsilon_n)}{\Gamma(\mathrm{i}\varepsilon_n - l)}\right)^{1/2} = \mathrm{i}^l(r/an)^l\left(\frac{1}{n^4a^3}\right)^{1/2}$$

we can reduce the radial wavefunction (4.14) to the standard flat-space result with a correct normalisation factor (see, e.g., [16]):

$$R_{nl}(r) = \left[\left(\frac{2}{na}\right)^3\frac{(n+l)!}{2n(n-l-1)!}\right]^{1/2}\frac{(2r/na)^l}{(2l+1)!}\exp(-r/na)\,{}_1F_1(1-n+l; 2l+2; 2r/na). \tag{4.17}$$

For large $n = kR$ we have $\varepsilon_n = -1/ak$. Using relations similar to (4.16) we find the radial scattering wavefunctions by replacing the sum Σ_n by an integral $\int_0^\infty R\,\mathrm{d}k$:

$$R_{kl}(r) = \frac{C_{kl}}{(2l+1)!}(2kr)^l\,\mathrm{e}^{-\mathrm{i}kr}\,{}_1F_1(1+l+\mathrm{i}/ak; 2l+2; 2\mathrm{i}kr) \tag{4.18}$$

$$C_{kl} = (2k/\pi)^{1/2}|\Gamma(1+l-\mathrm{i}/ak)|\sinh^{1/2}(\pi/ak). \tag{4.19}$$

This coincides with the result obtained by the usual analytical continuation consideration (see, e.g., [17]) except for a constant factor $[\exp(2\pi/ak)-1]^{-1/2}$.

5. Concluding remarks

In the above, we have presented a method to solve the Coulomb problem in a uniformly curved space by path integration. Since this problem has been solved by other means [1-3], there is nothing new insofar as the solution is concerned. However, the present path integral treatment has brought about a couple of points worth reporting.

Firstly, the present calculation is inevitably related to the knowledge that the system has the SU(1, 1) dynamical group [3]. After the angular contributions are separated, the radial path integral is mapped onto the dynamical group manifold of SU(1, 1) and evaluated explicitly in terms of Euler angles. Thus, establishing the link between the dynamical group and the path integral, we have obtained the energy spectrum and the correctly normalised wavefunctions for the Coulomb system on a sphere.

Secondly, by taking the flat-space limit, we are also able to find the standard results for the hydrogen-like atom. Therefore we have found another way of treating the hydrogen-like atom by path integration, which is totally new. As the SU(1, 1) dynamical group is non-compact, it has not only discrete but also continuous representations. In a spherical system the continuous parts are completely suppressed. However, in the flat-space limit the continuous contribution is revived. In fact, by the flat-space limiting process we have derived the Coulomb energy spectra and the wavefunctions for the bound as well as the scattering states.

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References

[1] Schrödinger E 1940 *Proc. R. Irish Acad.* A **46** 9; 1941 *Proc. R. Irish Acad.* A **46** 183, A **47** 53
Infeld L 1941 *Phys. Rev.* **59** 737
[2] Stevenson A F 1941 *Phys. Rev.* **59** 842
[3] Barut A O and Wilson R 1985 *Phys. Lett.* **110A** 351
[4] Inomata A and Kayed M A 1985 *J. Phys. A: Math. Gen.* **18** 235; 1985 *Phys. Lett.* **108A** 9
[5] Junker G and Inomata A 1986 *Path Integrals from meV to MeV* ed M C Gutzwiller, A Inomata, J R Klauder and L. Streit (Singapore: World Scientific) p 315
[6] Junker G and Böhm M 1986 *Phys. Lett.* **117A** 375
Böhm M and Junker G 1987 *J. Math. Phys.* to be published
[7] Inomata A and Wilson R 1986 *Conformal Groups and Related Symmetries* (*Lecture Notes in Physics* **261**) ed A O Barut and H D Doebner (Berlin: Springer) p 42; 1986 *Symmetries in Science II* ed B Gruber and R Lenczewski (New York: Plenum) p 255
[8] Cai J M, Cai P Y and Inomata A 1986 *Phys. Rev.* A **34** 4621
[9] Duru I H and Kleinert H 1979 *Phys. Lett.* **84B** 185; 1982 *Fortschr. Phys.* **30** 401
[10] Ho R and Inomata A 1982 *Phys. Rev. Lett.* **48** 231
Inomata A 1984 *Phys. Lett.* **101A** 253
[11] Inomata A 1986 *Path Integrals from meV to MeV* ed M C Gutzwiller, A Inomata, J R Klauder and L Streit (Singapore: World Scientific) p 433
[12] Gutzwiller M C 1985 *Phys. Scr.* T **9** 184
[13] Peak D and Inomata A 1969 *J. Math. Phys.* **10** 1422
[14] Bargmann V 1947 *Ann. Math.* **48** 568
[15] Barut A O and Phillips E C 1968 *Commun. Math. Phys.* **8** 52
[16] Merzbacher E 1970 *Quantum Mechanics* (New York: Wiley) 2nd edn, p 209
[17] Landau L D and Lifshitz E M 1958 *Quantum Mechanics, Non-Relativistic Theory* (Oxford: Pergamon) p 125